澳門大學 UNIVERSIDADE DE MACAU UNIVERSITY OF MACAU
澳門理工學院 Instituto Politécnico de Macau Macao Polytechnic Institute
旅遊學院 INSTITUTO DE FORMAÇÃO TURÍSTICA Institute for Tourism Studies
澳門科技大學 UNIVERSIDADE DE CIÊNCIA E TECNOLOGIA DE MACAU MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY

澳門四高校聯合入學考試（語言科及數學科）

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

模擬試題及參考答案 Mock Paper and Suggested Answer

數學附加卷 Mathematics Supplementary Paper

澳 門 四 高 校 聯 合 入 學 考 試

數 學 附 加 卷 模 擬 試 題

日期：二〇一Ｘ年M月Ｄ日　　限時：1 小時

指示：

1. 本卷有五條解答題，每題佔二十分。任擇三題作答。全卷滿分為六十分。

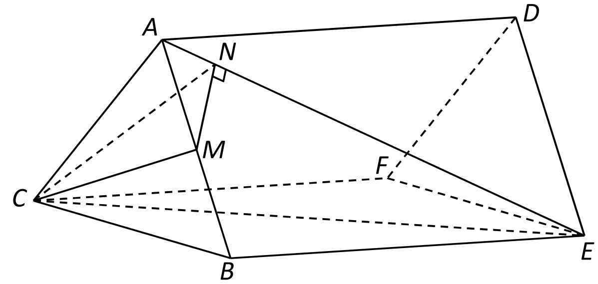
2. 如作答多於三題，只有首三題可得分。

3. 將所有答案寫在答案簿上。

4. 可用計算器，但不准用字典。

—本 卷 共 三 頁（包 括 本 頁）—

1. 在下圖中，直三棱柱 *ABC*–*DEF* 的側棱長為 2，*ABC* 是等邊三角形，其邊長為 1。設 *M* 為 *AB* 的中點，*N* 為 *AE* 上一點，且 $MN \perp AE$。



(a) 證明 $CM \perp AE$。 (4 分)

(b) 證明 ΔAMN 和 ΔAEB 是相似三角形。從而求 $|MN|$。 (5 分)

(c) 求三棱錐 *C*–*AMN* 的體積。 (5 分)

(d) 求面 *ACE* 與面 *ABE* 所成的二面角，答案以 arctan 表示。 (6 分)

2. (a) 因式分解行列式 $\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}$。 (6 分)

(b) 設 *a* 為實數。已知以 *x*、 *y*、*z* 為未知量的方程組： (8 分)

$$(E) \quad \begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases}$$

求 *a* 的各值使得方程組 (*E*)

(i) 有唯一解；

(ii) 有無限多解；

(iii) 無解。

(c) 設 $a = 2$。解方程組 (*E*)。 (6 分)

3. (a) 一底半徑為 x (> 0) 的正圓柱，其體積為 54π。設該圓柱的表面面積 (包括上、下兩底) 為 $S(x)$。

(i) 證明 $S(x)=2\pi\left(x^2+\dfrac{54}{x}\right)$。 (3 分)

(ii) 求 $\dfrac{dS}{dx}$ 及 $\dfrac{d^2S}{dx^2}$ 。 (2 分)

(iii)繪出曲綫 $y=S(x)$。在圖中 (如有的話) 把局部極大點、局部極小點和拐點標示出來。 (6 分)

(iv) 若 $1\le x\le 6$，求 $S(x)$ 的最大值。 (1 分)

(b) 求由曲綫 $y=5-x^2$ 與直綫 $y=x-1$ 所包圍的區域的面積。 (8 分)

4. 已知拋物綫 P: $y=4x^2$。設 $A(a, 4a^2)$ 為 P 上一點，其中 $a\neq 0$。

(a) 若直綫 $y=mx+c$ 與拋物綫 P 相切，證明 $m^2+16c=0$。由此，或用其他方法，推導出拋物綫 P 在點 A 的切綫的斜率為 $8a$。 (6 分)

(b) 設拋物綫 P 在點 A 的切綫及法綫分別與 y-軸相交於點 H 和點 K。證明 HK 的中點為點 $F\left(0, \dfrac{1}{16}\right)$。 (7 分)

(c) 設 C 是以點 F 為圓心並通過點 A 的圓。證明拋物綫 P 在點 A 的切綫與圓 C 在點 A 的切綫的夾角為 $\tan^{-1}|8a|$。 (7 分)

5. (a) 設 n 為正整數，$S_n=x+2x^2+\ldots+nx^n$，其中 $x\neq 1$。從考慮 S_n-xS_n 出發，證明

$$S_n=\frac{x(1-x^n)}{(1-x)^2}-\frac{nx^{n+1}}{1-x} 。$$

(5 分)

(b) 設 $\{a_n\}_{n=1}^{\infty}$ 為一等比數列，其公比 $r>1$，且有 $a_1+a_2+a_3=21$ 及 $a_1a_2a_3=216$。

(i) 求此等比數列的通項 a_n。 (6 分)

(ii) 用 (a) 的結果，求 $a_1^{a_1}a_2^{a_2}\cdots a_n^{a_n}$ ，$n=1, 2, \ldots$。 (9 分)

Joint Admission Examination for Four Higher Education Institutions in Macao

Mathematics Supplementary Paper Mock Questions

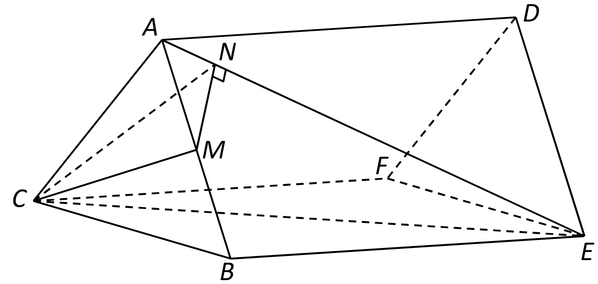
Date: dd/mm/201x　　　　　　　　　　　　　　Time Allowed: 1 Hour

Instructions:

1. There are five questions in this paper, each carries 20 marks. **Answer any three questions**. This paper carries 60 marks.
2. If more than three questions are answered, only the first three will be marked.
3. Put your answers in the answer book provided.
4. No dictionaries are allowed to be used. Calculators may be used.

── This paper consists of 3 pages (including this page) ──

1. In the diagram below, $ABC–DEF$ is a right triangular prism with $|AD|=2$. ABC is an equilateral triangle with side length 1. Let M be the midpoint of AB, N be the point on AE such that $MN \perp AE$.



(a) Show that $CM \perp AE$. (4 marks)

(b) Show that ΔAMN and ΔAEB are similar. Hence find $|MN|$. (5 marks)

(c) Find the volume of the triangular pyramid $C–AMN$. (5 marks)

(d) Find the angle between the plane ACE and the plane ABE. Give your answer in terms of arctan. (6 marks)

2. (a) Factorize the determinant $\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}$. (6 marks)

(b) Let a be a real number. Given the system of equations in unknowns x, y, z: (8 marks)

$$(E) \quad \begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases}$$

Find the value of a such that the system of equations (E)

(i) has a unique solution;

(ii) has infinitely many solutions;

(iii) has no solutions.

(c) Suppose $a=2$. Solve the system of equations (E). (6 marks)

3. (a) A right circular cylinder has base radius x (>0) and volume 54π. Let $S(x)$ be the surface area (including the two bases) of the cylinder.

   (i) Show that $S(x) = 2\pi\left(x^2 + \frac{54}{x}\right)$. (3 marks)

   (ii) Find $\frac{dS}{dx}$ and $\frac{d^2S}{dx^2}$. (2 marks)

   (iii) Sketch the curve $y = S(x)$. Mark in the graph (if any) the local maximum point, local minimum point, and inflection point. (6 marks)

   (iv) If $1 \leq x \leq 6$, find the maximum value of $S(x)$. (1 mark)

   (b) Find the area of the region bounded by the curve $y = 5 - x^2$ and the line $y = x - 1$. (8 marks)

4. Given parabola P: $y = 4x^2$. Let $A(a, 4a^2)$ be a point on P, where $a \neq 0$.

   (a) If the straight line $y = mx + c$ is tangent to P, show that $m^2 + 16c = 0$. Hence, or otherwise, deduce that the slope of the tangent line of P at A is $8a$. (6 marks)

   (b) Suppose that the tangent line and normal line of P at A intersect the y-axis at points H and K respectively. Show that the mid-point of HK is $F\left(0, \frac{1}{16}\right)$. (7 marks)

   (c) Let C be the circle centered at F and passing through A. Show that the angle between the tangent line of P at A and the tangent line of C at A is $\tan^{-1}|8a|$. (7 marks)

5. (a) Let n be a positive integer and $S_n = x + 2x^2 + \ldots + nx^n$, where $x \neq 1$. By considering $S_n - xS_n$, show that $S_n = \frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}$. (5 marks)

   (b) Let $\{a_n\}_{n=1}^{\infty}$ be a geometric progression with common ratio $r > 1$, and suppose $a_1 + a_2 + a_3 = 21$ and $a_1 a_2 a_3 = 216$.

   (i) Find the general term a_n of the geometric progression. (6 marks)

   (ii) Using the result in (a), find $a_1^{a_1} a_2^{a_2} \cdots a_n^{a_n}$, $n = 1, 2, \ldots$. (9 marks)

澳 門 四 高 校 聯 合 入 學 考 試

數 學 附 加 卷 模 擬 試 題 暨 參 考 答 案

日期：二〇一X年M月D日　　　　限時：1小時

指示：

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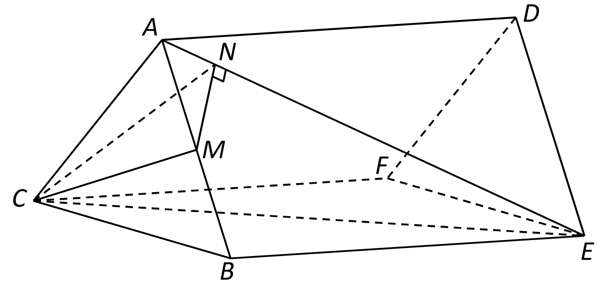
4. 可用計算器，但不准用字典。

—本 卷 共 七 頁（包 括 本 頁）—

1. 在下圖中，直三棱柱 *ABC*–*DEF* 的側棱長為 2，*ABC* 是等邊三角形，其邊長為 1。設 *M* 為 *AB* 的中點，*N* 為 *AE* 上一點，且 $MN \perp AE$。



(a) 證明 $CM \perp AE$。 (4 分)

(b) 證明 ΔAMN 和 ΔAEB 是相似三角形。從而求 $|MN|$。 (5 分)

(c) 求三棱錐 *C*–*AMN* 的體積。 (5 分)

(d) 求面 *ACE* 與面 *ABE* 所成的二面角，答案以 arctan 表示。 (6 分)

答案:(a) 由 *ABC* 是等邊三角形及 *M* 為 *AB* 的中點，得知 $CM \perp AB$。這與 $ABC \perp ABED$ 相結合，有 $CM \perp ABED$。因此得出 $CM \perp AE$。

(b) 由 $\angle MAN$ 是公共角，及 $\angle ABE$ 和 $\angle ANM$ 皆為直角，得知 ΔAMN 和 ΔAEB 是相似三角形。 $\therefore |MN| = \frac{|BE|}{|AE|} \cdot |AM| = \frac{2}{\sqrt{5}} \cdot \frac{1}{2} = \frac{1}{\sqrt{5}}$。

(c) $\because CM$ 為三棱錐的高，$\therefore$ 所求體積$= \frac{1}{3} \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{5}} \cdot \frac{1}{2\sqrt{5}} = \frac{\sqrt{3}}{120}$。

(d) 先證明 $CN \perp AE$。 $\therefore$ 所求的二面角$= \angle CNM = \tan^{-1}\left(\frac{\sqrt{3}}{2} \Big/ \frac{1}{\sqrt{5}}\right) = \tan^{-1}\frac{\sqrt{15}}{2}$。

2. (a) 因式分解行列式 $\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}$。 (6 分)

(b) 設 a 為實數。已知以 x、 y、z 為未知量的方程組： (8 分)

$$(E) \quad \begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases}$$

求 a 的各值使得方程組 (E)

(i) 有唯一解；

(ii) 有無限多解；

(iii) 無解。

(c) 設 $a=2$。解方程組 (E)。 (6 分)

答案: (a) 利用行和列的運算，得

$$\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = \begin{vmatrix} a & b & b \\ b & a & b \\ 0 & b-a & a-b \end{vmatrix} = \begin{vmatrix} a & 2b & b \\ b & a+b & b \\ 0 & 0 & a-b \end{vmatrix} = (a-b)\begin{vmatrix} a & 2b \\ b & a+b \end{vmatrix}$$

$$= (a-b)\begin{vmatrix} a-b & b-a \\ b & a+b \end{vmatrix} = (a-b)^2\begin{vmatrix} 1 & -1 \\ b & a+b \end{vmatrix} = (a-b)^2(a+2b)$$

(b) (E) 的係數矩陣為 $\begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}$。

由 (a) 可知其行列式 D 為 $D=(a-1)^2(a+2)$。

(i) 若 $a\neq 1$ 及 $a\neq -2$，則 $D\neq 0$。這情況下，(E) 有唯一解。

(ii) 若 $a=1$，(E) 中的三個方程是相同的，從而它有無限個解。

(iii) 若 $a=-2$，將 (E) 中的三個方程相加，得 $0=3$。這表示 (E) 無解。

(c) 當 $a=2$，(E) 變成

$$(E)\quad \begin{cases} 2x+\ \ y+\ \ z=1 & \text{-----}\ (1) \\ \ x+2y+\ \ z=2 & \text{-----}\ (2) \\ \ x+\ \ y+2z=4 & \text{-----}\ (3) \end{cases}$$

$3\times(3)-(1)-(2)\Rightarrow 4z=9\Rightarrow z=\frac{9}{4}$。 $(2)-(3)\Rightarrow y=z-2=\frac{1}{4}$。 $(1)-(2)\Rightarrow x=y-1=-\frac{3}{4}$。

3. (a) 一底半徑為 x (> 0) 的正圓柱，其體積為 54π。設該圓柱的表面面積 (包括上、下兩底) 為 $S(x)$。

(i) 證明 $S(x)=2\pi\left(x^2+\frac{54}{x}\right)$。 (3 分)

(ii) 求 $\frac{dS}{dx}$ 及 $\frac{d^2S}{dx^2}$。 (2 分)

(iii)繪出曲綫 $y=S(x)$。在圖中 (如有的話) 把局部極大點、局部極小點和拐點標示出來。 (6 分)

(iv) 若 $1\le x\le 6$，求 $S(x)$ 的最大值。 (1 分)

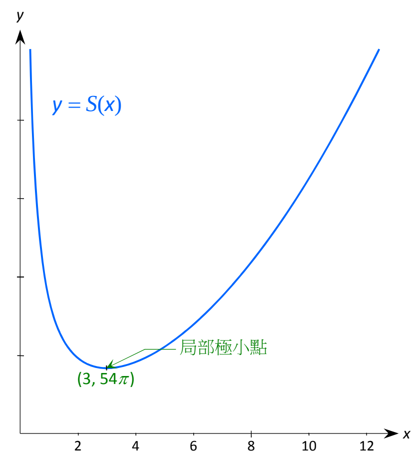
(b) 求由曲綫 $y=5-x^2$ 與直綫 $y=x-1$ 所包圍的區域的面積。 (8 分)

答案: (a) (i) 設 h 為圓柱體的高。若以 x 和 h 來表達，該圓柱體的體積 V 及表面面積 S 可由下式給出：$V=\pi x^2h$ 及 $S=2\pi x^2+2\pi xh$。

由已知條件得 $54\pi=\pi x^2h\Rightarrow xh=\frac{54}{x}$。$\therefore S(x)=2\pi x^2+2\pi\cdot\frac{54}{x}=2\pi\left(x^2+\frac{54}{x}\right)$。

(ii) $\because S(x)=2\pi(x^2+54x^{-1})$，$\therefore \frac{dS}{dx}=2\pi\left(2x-54x^{-2}\right)=4\pi\left(x-27x^{-2}\right)=4\pi\frac{x^3-27}{x^2}$。

$\therefore\ \dfrac{d^2S}{dx^2}=4\pi\left(1+54x^{-3}\right)=4\pi\dfrac{x^3+54}{x^3}$ 。



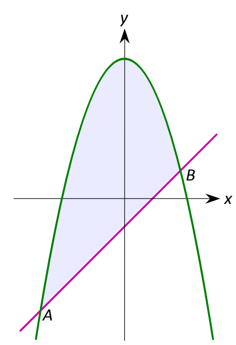
(iii) 從 (ii) 可以看出在 (0, 3) 上 $S'(x)<0$，而在 (3, +∞) 上 $S'(x)>0$。∴ $S(x)$ 在 (0, 3) 上下降而在 (3, +∞) 上上升。由此推出該曲綫在 $x=3$ 處有局部極小點，並且沒有局部極大點 (∵ $S(x)\to+\infty$ 當 $x\to0^+$ or $+\infty$)。

從 (ii) 亦可以看出在 (0, +∞) 上 $S''(x)>0$。由此推出該曲綫在整個區間 (0, +∞) 上向上凹，因而它沒有拐點。

$y=S(x)$ 的曲線繪畫在右圖。

(iv) 因為 $S(1)=110\pi$ 而 $S(6)=90\pi$，從上圖可知 110π 是 $S(x)$ 在 [1, 6] 上的最大值。

(b) 解聯立方程 $y=5-x^2$ 與 $y=x-1$，得知題中的直綫與拋物綫的交點為 $A(-3,-4)$ 和 $B(2, 1)$。



$$\therefore \text{所求面積}=\int_{-3}^{2}\left(\left(5-x^2\right)-\left(x-1\right)\right)dx$$

$$=\int_{-3}^{2}(6-x-x^2)\,dx$$

$$=\left[6x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-3}^{2}$$

$$=\frac{125}{6}$$

4. 已知拋物綫 P: $y=4x^2$。設 $A(a, 4a^2)$ 為 P 上一點，其中 $a\neq0$。

(a) 若直綫 $y=mx+c$ 與拋物綫 P 相切，證明 $m^2+16c=0$。由此，或用其他方法，推導出拋物綫 P 在點 A 的切綫的斜率為 $8a$。 (6 分)

(b) 設拋物綫 P 在點 A 的切綫及法綫分別與 y-軸相交於點 H 和點 K。證明 HK 的中點為點 $F\left(0, \frac{1}{16}\right)$。 (7 分)

(c) 設 C 是以點 F 為圓心並通過點 A 的圓。證明拋物綫 P 在點 A 的切綫與圓 C 在點 A 的切綫的夾角為 $\tan^{-1}|8a|$。 (7 分)

答案: (a) 直綫 $y=mx+c$ 與 P 相切於 A 當且僅當 $4x^2-mx-c=0$ 的判別式為 0。即 $(-m)^2-4\cdot 4(-c)=0 \Leftrightarrow m^2+16c=0$。

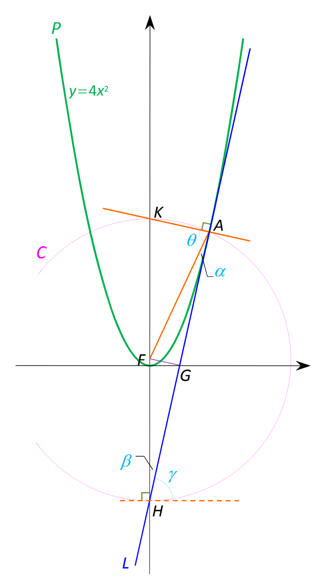
設 L: $y=mx+c$ 為 P 在 A 的切綫。則 m 為所求之斜率並有 $16c=-m^2$。因為 L 穿過 A，所以 $4a^2=ma+c$。$\therefore$ $64a^2=16ma+16c \Leftrightarrow (m-8a)^2=0 \Leftrightarrow m=8a$。

(b) 從 (a) 知 H 實為點 $(0, c)$，其中 $c=-\frac{1}{16}m^2=-4a^2$。

設 K 為點 $(0, d)$。由於 KA 是 P 在 A 的法綫，其斜率為 $-1/(8a)$。$\therefore$ $\frac{d-4a^2}{0-a}=\frac{-1}{8a} \Rightarrow d=4a^2+\frac{1}{8}$。

$\therefore$ 中點 F 的 y 座標為 $\frac{1}{2}(-4a^2+4a^2+\frac{1}{8})=\frac{1}{16}$。即 F 實為點 $\left(0, \frac{1}{16}\right)$。

(c) <u>方法一</u>



P 在 A 的切綫的斜率為 $m_1=8a$，而 C 在 A 的切綫的斜率為 $m_2=\frac{16a}{1-64a^2}$。此兩直綫的夾角為 $\tan^{-1}\left|\frac{m_1-m_2}{1+m_1m_2}\right|$。簡化後可得出結果。

<u>方法二</u>

由於 P 對稱於 y 軸，不妨假設 $a>0$ (見右圖)。

所要求的夾角相等於 P 在 A 的法綫 (即 KA) 與 C 在 A 的法綫 (即 FA) 之間的夾角。換句話說，這相當於求 θ。

設 α、β、γ 為圖中所示的角度，其中 γ 是 L 與 C 在 H 的切綫之間的夾角。

由於 K 的 y 座標為 $4a^2$ 而 H 的 y 座標為 $-4a^2$，G 是 HA 的中點。由此以及 F 是 HK 的中

點，可以推出 $FG \perp HA$。從而有 $FA=FH \Rightarrow \alpha=\beta$。

$\therefore\ \theta=90°-\alpha=90°-\beta=\gamma \Rightarrow \tan\theta=\tan\gamma=L$ 的斜率$=8a \Rightarrow \theta=\tan^{-1}(8a)$。

在以上論證中， 若 $a<0$，可以用 $-a$ 代替 a。

總括而言，所求之夾角為 $\tan^{-1}|8a|$。

5. (a) 設 n 為正整數，$S_n=x+2x^2+\ldots+nx^n$，其中 $x\neq 1$。從考慮 S_n-xS_n 出發，證明

$$S_n=\frac{x(1-x^n)}{(1-x)^2}-\frac{nx^{n+1}}{1-x}\text{。}$$ (5 分)

(b) 設 $\{a_n\}_{n=1}^{\infty}$ 為一等比數列，其公比 $r>1$，且有 $a_1+a_2+a_3=21$ 及 $a_1a_2a_3=216$。

(i) 求此等比數列的通項 a_n。 (6 分)

(ii) 用 (a) 的結果，求 $a_1^{a_1}a_2^{a_2}\cdots a_n^{a_n}$，$n=1, 2, \ldots$。 (9 分)

答案: (a) $S_n-xS_n=x+2x^2+\cdots+nx^n-x^2-2x^3-\cdots-nx^{n+1}=x+x^2+\cdots+x^n-nx^{n+1}=x\dfrac{(1-x^n)}{1-x}-nx^{n+1}$。

$$\therefore\ S_n=\frac{x(1-x^n)}{(1-x)^2}-\frac{nx^{n+1}}{1-x}\text{。}$$

(b) (i) 設 a 代表等比數列的首項。那麼 $a_n=ar^{n-1}$。

題中的兩個方程可簡化為 $a+ar+ar^2=21$ 與 $ar=6$。解這兩個方程得 $r=2$ 及 $a=3$。$\therefore$ $a_n=3\cdot 2^{n-1}$。

(ii) 首先有 $a_i^{a_i}=3^{3\cdot 2^{i-1}}\cdot 2^{3(i-1)2^{i-1}}$。

其次，將 $x=2$ 代入 (a) 中，可知 $2+2\cdot 2^2+\cdots+n\cdot 2^n=(n-1)\cdot 2^{n+1}+2$ 對所有正整數 n 成立。

$$\therefore\ a_1^{a_1}a_2^{a_2}\cdots a_n^{a_n}=3^{3(1+2+\cdots+2^{n-1})}\cdot 2^{3(2+2\cdot 2^2+\cdots+(n-1)\cdot 2^{n-1})}=27^{2^n-1}\cdot 8^{(n-2)\cdot 2^n+2}\text{。}$$

Joint Admission Examination for Four Higher Education Institutions in Macao

Mathematics Supplementary Paper Mock Questions and Suggested Solutions

Date: dd/mm/201x Time Allowed: 1 Hour

Instructions:

1. There are five questions in this paper, each carries 20 marks. **Answer any three questions**. This paper carries 60 marks.

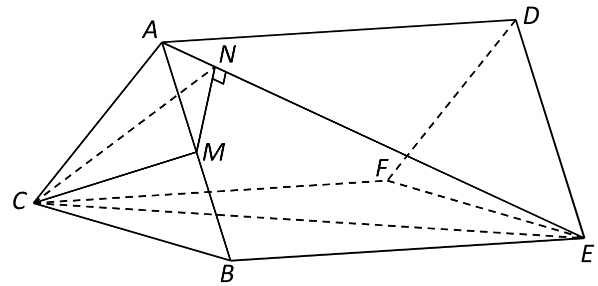
2. If more than three questions are answered, only the first three will be marked.

3. Put your answers in the answer book provided.

4. No dictionaries are allowed to be used. Calculators may be used.

— This paper consists of 7 pages (including this page) —

1. In the diagram below, *ABC*–*DEF* is a right triangular prism with $|AD|=2$. *ABC* is an equilateral triangle with side length 1. Let *M* be the midpoint of *AB*, *N* be the point on *AE* such that $MN \perp AE$.



(a) Show that $CM \perp AE$. (4 marks)

(b) Show that ΔAMN and ΔAEB are similar. Hence find $|MN|$. (5 marks)

(c) Find the volume of the triangular pyramid *C*–*AMN*. (5 marks)

(d) Find the angle between the plane *ACE* and the plane *ABE*. Give your answer in terms of arctan. (6 marks)

Ans.: (a) ΔABC is equilateral and *M* is the mid-point of *AB* imply $CM \perp AB$. Together with the fact that $ABC \perp ABED$, we have $CM \perp ABED$. Hence $CM \perp AE$.

(b) $\angle MAN$ is a common angle, and $\angle ABE$ and $\angle ANM$ are right angles. Hence ΔAMN and ΔAEB are similar. $\therefore\ |MN| = \frac{|BE|}{|AE|} \cdot |AM| = \frac{2}{\sqrt{5}} \cdot \frac{1}{2} = \frac{1}{\sqrt{5}}$.

(c) $\because$ *CM* is the height of the pyramid, $\therefore$ the required volume$= \frac{1}{3} \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{5}} \cdot \frac{1}{2\sqrt{5}} = \frac{\sqrt{3}}{120}$.

(d) Firstly, prove that $CN \perp AE$. $\therefore$ the required angle$=\angle CNM = \tan^{-1}\left(\frac{\sqrt{3}}{2} \Big/ \frac{1}{\sqrt{5}}\right) = \tan^{-1}\frac{\sqrt{15}}{2}$.

2. (a) Factorize the determinant $\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}$. (6 marks)

(b) Let *a* be a real number. Given the system of equations in unknowns *x*, *y*, *z*: (8 marks)

$$(E)\quad \begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases}$$

Find the value of a such that the system of equations (E)

(i) has a unique solution;

(ii) has infinitely many solutions;

(iii) has no solutions.

(c) Suppose $a=2$. Solve the system of equations (E). (6 marks)

Ans.: (a) Using row and column operations, we get

$$\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = \begin{vmatrix} a & b & b \\ b & a & b \\ 0 & b-a & a-b \end{vmatrix} = \begin{vmatrix} a & 2b & b \\ b & a+b & b \\ 0 & 0 & a-b \end{vmatrix} = (a-b)\begin{vmatrix} a & 2b \\ b & a+b \end{vmatrix}$$

$$= (a-b)\begin{vmatrix} a-b & b-a \\ b & a+b \end{vmatrix} = (a-b)^2\begin{vmatrix} 1 & -1 \\ b & a+b \end{vmatrix} = (a-b)^2(a+2b)$$

(b) The coefficient matrix of (E) is $\begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}$.

By (a), its determinant D is given by $D=(a-1)^2(a+2)$.

(i) If $a\neq 1$ and $a\neq -2$, then $D\neq 0$. In this case, (E) has a unique solution.

(ii) If $a=1$, the three equations in (E) are identical, and hence it has infinitely many solutions.

(iii) If $a=-2$, the three equations in (E) add up to $0=3$. This means that (E) has no solutions.

(c) When $a=2$, (E) becomes

$$(E)\quad \begin{cases} 2x + y + z = 1 & \text{-----}\ (1) \\ x + 2y + z = 2 & \text{-----}\ (2) \\ x + y + 2z = 4 & \text{-----}\ (3) \end{cases}$$

$3\times(3)-(1)-(2) \Rightarrow 4z=9 \Rightarrow z=\frac{9}{4}$. $(2)-(3) \Rightarrow y=z-2=\frac{1}{4}$. $(1)-(2) \Rightarrow x=y-1=-\frac{3}{4}$.

3. (a) A right circular cylinder has base radius x (>0) and volume 54π. Let $S(x)$ be the surface area (including the two bases) of the cylinder.

(i) Show that $S(x) = 2\pi\left(x^2 + \frac{54}{x}\right)$. (3 marks)

(ii) Find $\frac{dS}{dx}$ and $\frac{d^2S}{dx^2}$. (2 marks)

(iii) Sketch the curve $y = S(x)$. Mark in the graph (if any) the local maximum point, local minimum point, and inflection point. (6 marks)

(iv) If $1 \le x \le 6$, find the maximum value of $S(x)$. (1 mark)

(b) Find the area of the region bounded by the curve $y = 5 - x^2$ and the line $y = x - 1$. (8 marks)

Ans.: (a) (i) Let h be the height of the cylinder. In terms of x and h, the volume V and surface area S of the cylinder are given by $V = \pi x^2 h$ and $S = 2\pi x^2 + 2\pi xh$.

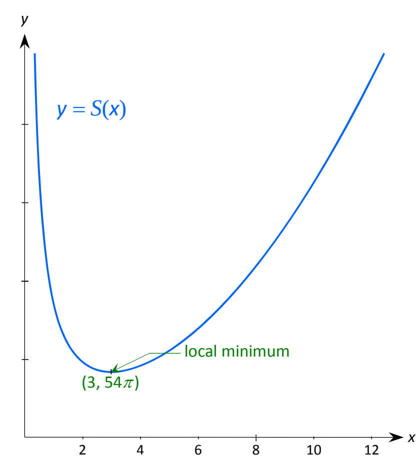
From the given, $54\pi = \pi x^2 h \Rightarrow xh = \frac{54}{x}$. $\therefore$ $S(x) = 2\pi x^2 + 2\pi \cdot \frac{54}{x} = 2\pi\left(x^2 + \frac{54}{x}\right)$.

(ii) $\because$ $S(x) = 2\pi(x^2 + 54x^{-1})$, $\therefore$ $\frac{dS}{dx} = 2\pi\left(2x - 54x^{-2}\right) = 4\pi\left(x - 27x^{-2}\right) = 4\pi\frac{x^3 - 27}{x^2}$.

$\therefore$ $\frac{d^2S}{dx^2} = 4\pi\left(1 + 54x^{-3}\right) = 4\pi\frac{x^3 + 54}{x^3}$.

(iii) From (ii), we see that $S'(x) < 0$ on $(0, 3)$, and $S'(x) > 0$ on $(3, +\infty)$. $\therefore$ $S(x)$ is decreasing on $(0, 3)$, and increasing on $(3, +\infty)$. This implies that the curve has a local minimum point at $x = 3$, and it has no local maximum point ($\because$ $S(x) \to +\infty$ as $x \to 0^+$ or $+\infty$).

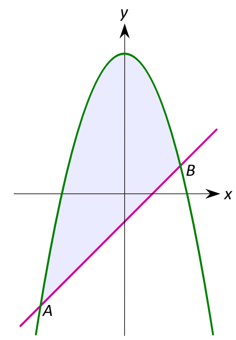
We see from (ii) also that $S''(x) > 0$ on $(0, +\infty)$. This, implies that the curve is concave up throughout $(0, +\infty)$, and hence it has no inflection point.



The curve $y = S(x)$ is sketched on the right.

(iv) Since $S(1) = 110\pi$ and $S(6) = 90\pi$, we conclude from the graph of $y = S(x)$ that 110π is the maximum value of $S(x)$ on $[1, 6]$.

(b) Solving the equations $y = 5 - x^2$ and $y = x - 1$ simultaneously, we know that the line intersects the parabola at $A(-3,-4)$ and $B(2, 1)$.



$\therefore$ The required area $= \int_{-3}^{2} \left((5 - x^2) - (x - 1)\right) dx$

$= \int_{-3}^{2} (6 - x - x^2)\, dx$

$= \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$

$= \frac{125}{6}$

4. Given parabola P: $y = 4x^2$. Let $A(a, 4a^2)$ be a point on P, where $a \neq 0$.

(a) If the straight line $y = mx + c$ is tangent to P, show that $m^2 + 16c = 0$. Hence, or otherwise, deduce that the slope of the tangent line of P at A is $8a$. (6 marks)

(b) Suppose that the tangent line and normal line of P at A intersect the y-axis at points H and K respectively. Show that the mid-point of HK is $F\left(0, \frac{1}{16}\right)$. (7 marks)

(c) Let C be the circle centered at F and passing through A. Show that the angle between the tangent line of P at A and the tangent line of C at A is $\tan^{-1}|8a|$. (7 marks)

Ans.: (a) The line $y = mx + c$ is tangent to P at A if and only if the discriminant of $4x^2 - mx - c = 0$ is 0. That is, $(-m)^2 - 4\cdot 4(-c) = 0 \Leftrightarrow m^2 + 16c = 0$.

Let L: $y = mx + c$ be a tangent to P at A. Then m is the required slope and $16c = -m^2$. Since L passes through A, we have $4a^2 = ma + c$. $\therefore$ $64a^2 = 16ma + 16c \Leftrightarrow (m - 8a)^2 = 0 \Leftrightarrow m = 8a$.

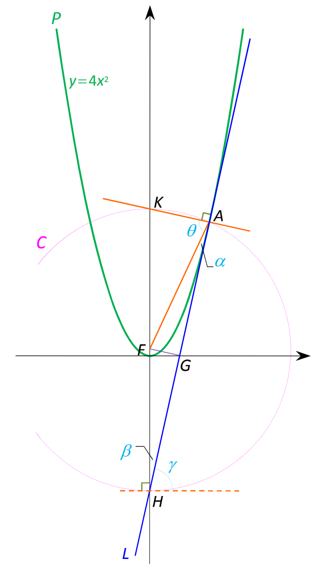
(b) From (a), H is the point $(0, c)$, where $c = -\frac{1}{16}m^2 = -4a^2$.

Let K be the point $(0, d)$. Since KA is normal to P at A, its slope is $-1/(8a)$. $\therefore$ $\frac{d - 4a^2}{0 - a} = \frac{-1}{8a} \Rightarrow d = 4a^2 + \frac{1}{8}$.

$\therefore$ The y-coordinate of the mid-point F is $\frac{1}{2}\left(-4a^2 + 4a^2 + \frac{1}{8}\right) = \frac{1}{16}$, i.e., F is the point $\left(0, \frac{1}{16}\right)$.

(c) Method 1

Slope of the tangent of P at A is $m_1 = 8a$, and slope of the tangent of C at A is $m_2 = \frac{16a}{1-64a^2}$. The angle between the two lines is $\tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$. Upon simplification, the result follows.



Method 2

Since P is symmetrical about the y-axis, we may assume, without loss of generality, that $a > 0$ (see the right picture).

Observe that the required angle is the same as the angle between the normal of P at A (i.e. KA) and the normal of C at A (i.e. FA). Thus it is the same as to find the angle θ.

Let α, β, γ be the angles as shown in the picture, where γ is the angle between L and the tangent line of C at H.

Since the y-coordinate of A is $4a^2$ and the y-coordinate of H is $-4a^2$, G is the mid-point of HA. From this and the fact that F is the mid-point of HK, we infer that $FG \perp HA$. It follows that $FA = FH \Rightarrow \alpha = \beta$.

$\therefore\ \theta = 90° - \alpha = 90° - \beta = \gamma \Rightarrow \tan\theta = \tan\gamma = \text{slope of } L = 8a \Rightarrow \theta = \tan^{-1}(8a)$.

In the above argument, if $a < 0$, we may replace a by $-a$.

Summing up, the required angle is $\tan^{-1}|8a|$.

5. (a) Let n be a positive integer and $S_n = x + 2x^2 + \ldots + nx^n$, where $x \neq 1$. By considering $S_n - xS_n$, show that $S_n = \frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}$. (5 marks)

(b) Let $\{a_n\}_{n=1}^{\infty}$ be a geometric progression with common ratio $r > 1$, and suppose $a_1 + a_2 + a_3 = 21$ and $a_1 a_2 a_3 = 216$.

(i) Find the general term a_n of the geometric progression. (6 marks)

(ii) Using the result in (a), find $a_1^{a_1} a_2^{a_2} \cdots a_n^{a_n}$, $n = 1, 2, \ldots$. (9 marks)

Ans.: (a) $S_n - xS_n = x + 2x^2 + \cdots + nx^n - x^2 - 2x^3 - \cdots - nx^{n+1} = x + x^2 + \cdots + x^n - nx^{n+1} = x\frac{(1-x^n)}{1-x} - nx^{n+1}$.

$\therefore\ S_n = \frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}$.

(b) (i) Let a denote the first term of the geometric progression. Then $a_n = ar^{n-1}$.

The two given equations can be simplified as $a + ar + ar^2 = 21$ and $ar = 6$. Solving these two equations, we obtain $r = 2$ and $a = 3$. $\therefore$ $a_n = 3 \cdot 2^{n-1}$.

(ii) Firstly note that $a_i^{a_i} = 3^{3 \cdot 2^{i-1}} \cdot 2^{3 \cdot (i-1) 2^{i-1}}$.

Secondly, by setting $x = 2$ in (a), we have $2 + 2 \cdot 2^2 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2$ for all positive integers n.

$\therefore\ a_1^{a_1} a_2^{a_2} \cdots a_n^{a_n} = 3^{3(1+2+\cdots+2^{n-1})} \cdot 2^{3(2+2 \cdot 2^2+\cdots+(n-1) \cdot 2^{n-1})} = 27^{2^n - 1} \cdot 8^{(n-2) \cdot 2^n + 2}$.