澳門四高校聯合入學考試 (語言科及數學科)

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2017年試題及參考答案
2017 Examination Paper and Suggested Answer

數學正卷　Mathematics Standard Paper

<u>第一部分</u> 選擇題。每題佔四分，共佔六十分。

1. 設 $P=\{-2, a-2, 3\}$，$Q=\{1, a^2-a-3, -5\}$。若 $P\cap Q=\{3\}$，則

A. $a=2$　　B. $a=-2$　　C. $a=3$

D. $a=-2$ 或 $a=3$　　E. 以上皆非

2. 解不等式 $\dfrac{x+2}{3}<\dfrac{4x+5}{6}<\dfrac{7x+8}{9}$。

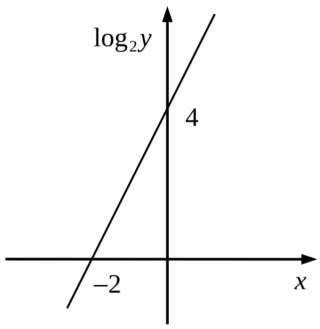
A. $x>2$　　B. $x>\dfrac{1}{2}$　　C. $x>-\dfrac{1}{2}$

D. $x<-\dfrac{1}{2}$　　E. $-\dfrac{1}{2}<x<\dfrac{1}{2}$

3. 把 5000 元存入銀行，每年計息一次，以複利計算，2 年後的本利和為 5408 元，則年利率為

A. 3.4%　　B. 4%　　C. 4.2%　　D. 3.6%　　E. 4.5%

4. x 與 $\log_2 y$ 之間的線性關係如下圖所示。若 $y=ab^x$，則 $a+b=$



A. 4　　B. 64　　C. $\dfrac{1}{4}$　　D. 12　　E. 20

5. 若 $x+\dfrac{1}{x}=6+\dfrac{1}{6}$，則 $x=$

A. 6　　B. $\dfrac{1}{6}$　　C. 6或$\dfrac{1}{6}$　　D. 6或$-\dfrac{1}{6}$　　E. -6或$\dfrac{1}{6}$

6. $\left(\dfrac{2\log_{10}2+\log_{10}6}{-\log_{10}3-3\log_{10}2}\right)^5-9^5\times3^{-11}=$

A. $\dfrac{3}{2}$　　B. $-\dfrac{3}{4}$　　C. $\dfrac{2}{3}$　　D. $-\dfrac{4}{3}$　　E. 0

7. 若 $x^3+3x-a-3$ 能被 $x-1$ 整除，則 $a=$

A. -1　　B. 1　　C. -2　　D. 2　　E. 以上皆非

8. 函數 $6(7-x)+[9-(7-x)]x$ 的最小值是

A. 36　　B. 42　　C. 38　　D. 34　　E. 以上皆非

9. 若 $\sin x - \cos x = \frac{1}{3}$，則 $\sin x \cos x =$

A. $\frac{2}{9}$ B. $-\frac{4}{9}$ C. $-\frac{2}{9}$ D. $-\frac{1}{3}$ E. 以上皆非

10. 方程 $\ln(x-1) + \ln x = \ln 6$ 有多少個實根？

A. 0 B. 1 C. 2 D. 3 E. 4

11. 已知 a_1, a_2, a_3, a_4 組成等比數列。若 a_1 和 a_4 是方程 $3x^2 - 5x + 2 = 0$ 的兩個根，則 $a_2 a_3 =$

A. $\frac{2}{3}$ B. $-\frac{3}{2}$ C. $\frac{5}{3}$ D. $-\frac{5}{3}$ E. $-\frac{2}{3}$

12. 指數函數 $y = e^x$ 的圖像先向上移動一個單位，然後再向右平移一個單位。新圖像的函數表達式是什麼？

A. $y = e^{x+1} + 1$ B. $y = e^{x-1} - 1$ C. $y = -e^x + 1$

D. $y = e^{x+1} - 1$ E. $y = e^{x-1} + 1$

13. 設 $\alpha \in (0, 2\pi)$。若 $\cos\alpha > 0$ 和 $\tan\alpha < 0$，則

A. $\alpha \in \left(0, \frac{\pi}{2}\right)$ B. $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ C. $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$

D. $\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$ E. $\alpha \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

14. 若 F_1、F_2 是橢圓 $\frac{x^2}{9} + \frac{y^2}{25} = 1$ 的兩個焦點，M 是橢圓上任意一點，則 ΔMF_1F_2 的周長為

A. 16 B. 18 C. 22 D. 20 E. 以上皆非

15. 在 $\left(x - \frac{1}{x}\right)^{10}$ 的展開式中，常數項是多少？

A. -252 B. 252 C. 0 D. -32 E. 32

<u>第二部分</u> 解答題。每題佔八分，共佔四十分。

1. 把 $\dfrac{x^3+x}{x^2-3x+2}$ 化為部分分式。 (8 分)

2. 圓 C 的圓心坐標為(2, 3)，x 軸為 C 的一條切線。直線 L 的斜率及 y 軸的截距分別為−1 及 b。
   (a) 求圓 C 的方程。 (2 分)
   (b) 求 b 的範圍使得 L 與 C 相交於不同的兩點 A 及 B。 (3 分)
   (c) 已知線段 AB 的長度 $|AB|$ 等於 2，求 b 的值。 (3 分)

3. 從 3 名男生和 4 名女生中隨機地選出 3 人參加歌唱比賽，求以下各事件的概率。
   (a) 3 人都是男生。 (2 分)
   (b) 1 名男生，2 名女生。 (2 分)
   (c) 2 名男生，1 名女生。 (2 分)
   (d) 3 人中至少有 1 名女生。 (2 分)

4. 等差數列 $\{a_n\}_{n\geq 1}$ 滿足 $a_1+a_2+a_3=0$ 和 $a_1+a_2+a_3+a_4+a_5=5$。
   (a) 求通項 a_n 的公式。 (4分)
   (b) 設 $b_n=\dfrac{1}{a_{n+3}a_{n+4}}$ $(n\geq 1)$，求數列 $\{b_n\}_{n\geq 1}$ 的前 n 項和。 (4分)

5. 用數學歸納法證明對於任一正整數 n，$3^{3n+1}-5^{n+2}$ 可被 22 整除。 (8 分)

參考答案

<u>第一部份</u> 選擇題。

題目編號	最佳答案
1	B
2	C
3	B
4	E
5	C
6	D
7	B
8	C
9	E
10	B
11	A
12	E
13	D
14	B
15	A

(第二部分答案由下頁開始)

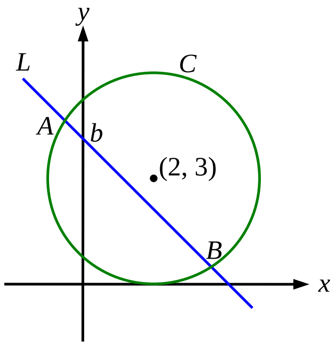
第二部份 **解答題。**

1. 利用長除法，得 $\dfrac{x^3+x}{x^2-3x+2}=x+3+\dfrac{8x-6}{x^2-3x+2}$。

設 $\dfrac{8x-6}{x^2-3x+2}=\dfrac{8x-6}{(x-1)(x-2)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}$。則 $8x-6=A(x-2)+B(x-1)$。

比較對應項係數得 $A=-2$，$B=10$。因此 $\dfrac{x^3+x}{x^2-3x+2}=x+3-\dfrac{2}{x-1}+\dfrac{10}{x-2}$。

2. (a) 據題意，可知半徑 $r=3$。$\therefore$ 圓 C 的方程為 $(x-2)^2+(y-3)^2=3^2$，即 $x^2+y^2-4x-6y+4=0$。



(b) **代數方法**

直線 L 的方程為 $y=-x+b$。

A、B 為方程組 $\begin{cases} y=-x+b \\ x^2+y^2-4x-6y+4=0 \end{cases}$ 的解。

消去 y 得 $2x^2-2(b-1)x+(b^2-6b+4)=0$ ------------------------ (1)

所求的 b 的範圍使得方程 (1) 的判別式 Δ 大於 0，即

$$(b-1)^2-2(b^2-6b+4)>0 \quad \Leftrightarrow \quad b^2-10b+7<0 \quad \Leftrightarrow \quad 5-3\sqrt{2}<b<5+3\sqrt{2}\text{。}$$

幾何方法

設 d 為圓心 $(2, 3)$ 到直線 L: $x+y-b=0$ 的距離，則 $d=\dfrac{|2+3-b|}{\sqrt{1+1}}=\dfrac{|5-b|}{\sqrt{2}}$。

L 與 C 相交於不同的兩點當且僅當 $d<r$，即 $\dfrac{|5-b|}{\sqrt{2}}<3$。

解以上不等式得 $5-3\sqrt{2}<b<5+3\sqrt{2}$。

(c) **代數方法**

設 (x_1, y_1)、(x_2, y_2) 分別為 A、B 的座標。$\because$ L 的斜率為 -1，$\therefore$ $|AB|=\sqrt{2}\,|x_2-x_1|$。

由韋達定理 (即根與係數的關係) 得 $|AB|^2=2(x_2-x_1)^2=2[(x_2+x_1)^2-4x_1x_2]=\Delta/2=-2(b^2-10b+7)$。

若 $|AB|=2$，則 $b^2-10b+7=-2 \Rightarrow b=1$ 或 9。經驗證，這兩個值都符合要求。

幾何方法

如上所述，圓心 $(2, 3)$ 到直線 L 的距離 $d=\dfrac{|2+3-b|}{\sqrt{1+1}}=\dfrac{|5-b|}{\sqrt{2}}$。

從勾股定理，得 $d^2+\left(\frac{1}{2}|AB|\right)^2=r^2$，即 $\left(\dfrac{|5-b|}{\sqrt{2}}\right)^2+1^2=3^2$。

解方程得 $b=1$ 或 9。經驗證，這兩個值都符合要求。

3. (a) 所求概率 $= {}_3C_3/{}_7C_3 = 1/35$。

(b) 所求概率 $= {}_3C_1\cdot{}_4C_2/{}_7C_3 = 18/35$。

(c) 所求概率 $= {}_3C_2\cdot{}_4C_1/{}_7C_3 = 12/35$。

(d) <u>方法一</u>：所求概率 $=1-$ 沒有女生的概率 $=1-1/35=34/35$ (見 (a) 的答案)。

<u>方法二</u>：所求概率 $=({}_4C_1\cdot{}_3C_2+{}_4C_2\cdot{}_3C_1+{}_4C_3)/{}_7C_3=34/35$。

4. (a) 設 a 和 d 分別為 $\{a_n\}_{n\geq 1}$ 的首項和公差。

據題意，$3a+3d=0$ 和 $5a+10d=5$。解這兩條方程，得 $a=-1$ 及 $d=1$。

$\therefore\ a_n=a+(n-1)d=n-2$。

(b) $b_n=\dfrac{1}{(n+1)(n+2)}=\dfrac{1}{n+1}-\dfrac{1}{n+2}$。

$\therefore\ b_1+b_2+\cdots+b_n=\left(\dfrac{1}{2}-\dfrac{1}{3}\right)+\left(\dfrac{1}{3}-\dfrac{1}{4}\right)+\cdots+\left(\dfrac{1}{n+1}-\dfrac{1}{n+2}\right)=\dfrac{1}{2}-\dfrac{1}{n+2}=\dfrac{n}{2(n+2)}$。

5. 設 $S(n)$ 代表命題 "$3^{3n+1}-5^{n+2}$ 可被 22 整除"。

由於 $3^4-5^3=-44=22\cdot(-2)$，$S(1)$ 成立。

假設 $S(k)$ 對某正整數 k 成立，即假定 $3^{3k+1}-5^{k+2}=22a$，a 為整數。

考慮 $S(k+1)$：$3^{3k+4}-5^{k+3}=27\cdot 3^{3k+1}-5\cdot 5^{k+2}=27(22a+5^{k+2})-5\cdot 5^{k+2}=22(27a+5^{k+2})$。

換句話說，$S(k+1)$ 也成立。

根據數學歸納法原理，可知 $S(n)$ 對所有正整數 n 都成立。

<u>Part 1</u> Multiple choice questions. Each question carries 4 marks with a total of 60 marks.

1. Let $P=\{-2, a-2, 3\}$, $Q=\{1, a^2-a-3, -5\}$. If $P\cap Q=\{3\}$, then

   A. $a=2$ B. $a=-2$ C. $a=3$

   D. $a=-2$ or $a=3$ E. none of the above

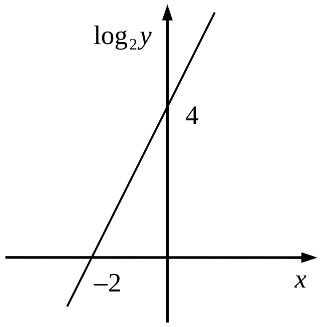
2. Solve the inequalities $\frac{x+2}{3}<\frac{4x+5}{6}<\frac{7x+8}{9}$.

   A. $x>2$ B. $x>\frac{1}{2}$ C. $x>-\frac{1}{2}$ D. $x<-\frac{1}{2}$ E. $-\frac{1}{2}<x<\frac{1}{2}$

3. Suppose that $5000 is deposited in a savings account and the interest rate is compounded annually. The total amount on deposit at the end of 2 years is $5408. What is the annual interest rate?

   A. 3.4% B. 4% C. 4.2% D. 3.6% E. 4.5%

4. The linear relationship between x and $\log_2 y$ is as shown in following figure. If $y=ab^x$, then $a+b=$



   A. 4 B. 64 C. $\frac{1}{4}$ D. 12 E. 20

5. If $x+\frac{1}{x}=6+\frac{1}{6}$, then $x=$

   A. 6 B. $\frac{1}{6}$ C. 6 or $\frac{1}{6}$ D. 6 or $-\frac{1}{6}$ E. -6 or $\frac{1}{6}$

6. $\left(\frac{2\log_{10}2+\log_{10}6}{-\log_{10}3-3\log_{10}2}\right)^5-9^5\times 3^{-11}=$

   A. $\frac{3}{2}$ B. $-\frac{3}{4}$ C. $\frac{2}{3}$ D. $-\frac{4}{3}$ E. 0

7. If $x^3+3x-a-3$ is divisible by $x-1$, then $a=$

   A. -1 B. 1 C. -2

   D. 2 E. none of the above

8. The minimum value of the function $6(7-x)+[9-(7-x)]x$ is

   A. 36 B. 42 C. 38

   D. 34 E. none of the above

9. If $\sin x - \cos x = \frac{1}{3}$, then $\sin x \cos x =$

A. $\frac{2}{9}$ B. $-\frac{4}{9}$ C. $-\frac{2}{9}$

D. $-\frac{1}{3}$ E. none of the above

10. How many real roots does the equation $\ln(x-1) + \ln x = \ln 6$ have?

A. 0   B. 1   C. 2   D. 3   E. 4

11. Suppose that a_1, a_2, a_3, a_4 form a geometric sequence. If a_1 and a_4 are the roots of the equation $3x^2 - 5x + 2 = 0$, then $a_2 a_3 =$

A. $\frac{2}{3}$ B. $-\frac{3}{2}$ C. $\frac{5}{3}$ D. $-\frac{5}{3}$ E. $-\frac{2}{3}$

12. The graph of the exponential function $y = e^x$ is shifted upward by one unit, and then shifted to the right by one unit. What is the equation of the new graph?

A. $y = e^{x+1} + 1$ B. $y = e^{x-1} - 1$ C. $y = -e^x + 1$

D. $y = e^{x+1} - 1$ E. $y = e^{x-1} + 1$

13. Suppose $\alpha \in (0, 2\pi)$. If $\cos\alpha > 0$ and $\tan\alpha < 0$, then

A. $\alpha \in \left(0, \frac{\pi}{2}\right)$ B. $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ C. $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$

D. $\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$ E. $\alpha \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$

14. Suppose that F_1 and F_2 are the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$, and that M is any point on the ellipse, then the perimeter of ΔMF_1F_2 is

A. 16   B. 18   C. 22

D. 20   E. none of the above

15. In the expansion of $\left(x - \frac{1}{x}\right)^{10}$, what is the constant term?

A. -252 B. 252   C. 0   D. -32 E. 32

<u>Part II</u> Problem-solving questions. Each question carries 8 marks with a total 40 marks.

1. Find the partial fraction decomposition of $\dfrac{x^3+x}{x^2-3x+2}$. (8 marks)

2. Let C be a circle centered at (2, 3) and x-axis be a tangent to C. Let L be a straight line with slope -1 and y-intercept b.
   (a) Find the equation of the circle C. (2 marks)
   (b) Find the range of b such that the straight line L and the circle C intersect at two different points A and B. (3 marks)
   (c) Given that $|AB|$, length of the line segment AB, equals 2. Find the value of b. (3 marks)

3. Suppose three persons are randomly chosen from three boys and four girls to compete in a singing contest. What is the probability for each of the following events?
   (a) All the three boys are chosen. (2 marks)
   (b) One boy and two girls are chosen. (2 marks)
   (c) Two boys and one girl are chosen. (2 marks)
   (d) At least one girl is chosen. (2 marks)

4. Let $\{a_n\}_{n\geq 1}$ be an arithmetic sequence such that $a_1+a_2+a_3=0$ and $a_1+a_2+a_3+a_4+a_5=5$.
   (a) Find the general term a_n of the arithmetic sequence. (4 marks)
   (b) Let $b_n=\dfrac{1}{a_{n+3}a_{n+4}}$ $(n\geq 1)$, find the sum of the first n terms of $\{b_n\}_{n\geq 1}$. (4 marks)

5. Prove by mathematical induction that $3^{3n+1}-5^{n+2}$ is divisible by 22 for any positive integer n. (8 marks)

Suggested Answer

Part I Multiple choice questions.

Question Number	Best Answer
1	B
2	C
3	B
4	E
5	C
6	D
7	B
8	C
9	E
10	B
11	A
12	E
13	D
14	B
15	A

(Answers for Part II start from next page)

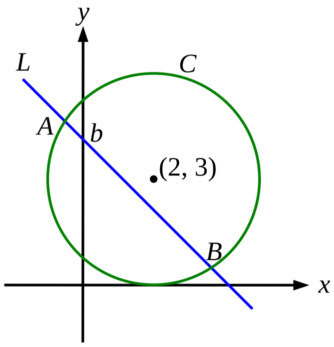
<u>Part II</u> Problem-solving questions.

1. By long division, we get $\dfrac{x^3+x}{x^2-3x+2}=x+3+\dfrac{8x-6}{x^2-3x+2}$.

   Let $\dfrac{8x-6}{x^2-3x+2}=\dfrac{8x-6}{(x-1)(x-2)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}$. Then $8x-6=A(x-2)+B(x-1)$.

   Comparing coefficients of both sides yields $A=-2$, $B=10$. Hence $\dfrac{x^3+x}{x^2-3x+2}=x+3-\dfrac{2}{x-1}+\dfrac{10}{x-2}$.

2. (a) The given implies that the radius $r=3$. $\therefore$ The equation of circle C is given by $(x-2)^2+(y-3)^2=3^2$, that is, $x^2+y^2-4x-6y+4=0$.



(b) **<u>Algebraic Method</u>**

The equation of line L is $y=-x+b$. A and B are solutions of the system $\begin{cases} y=-x+b \\ x^2+y^2-4x-6y+4=0 \end{cases}$.

Eliminating y yields $2x^2-2(b-1)x+(b^2-6b+4)=0$ ------------ (1)

For the required range of b, the discriminant Δ of (1) is greater than 0, i.e. $(b-1)^2-2(b^2-6b+4)>0$ $\Leftrightarrow$ $b^2-10b+7<0$ $\Leftrightarrow$ $5-3\sqrt{2}<b<5+3\sqrt{2}$.

<u>Geometric Method</u>

Let d be the distance from the center (2, 3) to the line L: $x+y-b=0$. Then $d=\dfrac{|2+3-b|}{\sqrt{1+1}}=\dfrac{|5-b|}{\sqrt{2}}$.

L intersects C at two distinct points if and only if $d<r$, i.e. $\dfrac{|5-b|}{\sqrt{2}}<3$.

Solving the inequality yields $5-3\sqrt{2}<b<5+3\sqrt{2}$.

(c) **<u>Algebraic Method</u>**

Let (x_1, y_1) and (x_2, y_2) be respectively the coordinates of points A and B. $\because$ The slope of L is -1, $\therefore$ $|AB|=\sqrt{2}\,|x_2-x_1|$.

By Viète Theorem (i.e., the relationship between roots and coefficients), we get

$$|AB|^2=2(x_2-x_1)^2=2[(x_2+x_1)^2-4x_1x_2]=\Delta/2=-2(b^2-10b+7).$$

If $|AB|=2$, then $b^2-10b+7=-2 \Rightarrow b=1$ or 9. Upon verification, both values are acceptable.

<u>Geometric Method</u>

As mentioned above, the distance d from (2, 3) to L is given by $d=\dfrac{|2+3-b|}{\sqrt{1+1}}=\dfrac{|5-b|}{\sqrt{2}}$.

From Pythagoras Theorem, we get $d^2+\left(\frac{1}{2}|AB|\right)^2=r^2$, i.e. $\left(\dfrac{|5-b|}{\sqrt{2}}\right)^2+1^2=3^2$.

Solving the equation yields $b=1$ or 9. Upon verification, both values are acceptable.

3. (a) Required probability $= {}_3C_3/{}_7C_3 = 1/35$.

   (b) Required probability $= {}_3C_1 \cdot {}_4C_2/{}_7C_3 = 18/35$.

   (c) Required probability $= {}_3C_2 \cdot {}_4C_1/{}_7C_3 = 12/35$.

   (d) Method 1: Required probability $= 1-$"probability of no girls"$=1-1/35=34/35$ (see the answer of (a)).

   Method 2: Required probability $= ({}_4C_1 \cdot {}_3C_2 + {}_4C_2 \cdot {}_3C_1 + {}_4C_3)/{}_7C_3 = 34/35$.

4. (a) Let a and d be respectively the first term and the common difference of $\{a_n\}_{n\geq 1}$.

   The given implies that $3a+3d=0$ and $5a+10d=5$. Solving these equations yields $a=-1$ and $d=1$.

   $\therefore\ a_n=a+(n-1)d=n-2$.

   (b) $b_n = \dfrac{1}{(n+1)(n+2)} = \dfrac{1}{n+1} - \dfrac{1}{n+2}$.

   $\therefore\ b_1+b_2+\cdots+b_n = \left(\dfrac{1}{2}-\dfrac{1}{3}\right)+\left(\dfrac{1}{3}-\dfrac{1}{4}\right)+\cdots+\left(\dfrac{1}{n+1}-\dfrac{1}{n+2}\right)=\dfrac{1}{2}-\dfrac{1}{n+2}=\dfrac{n}{2(n+2)}$.

5. Let $S(n)$ denote the statement "$3^{3n+1}-5^{n+2}$ is divisible by 22".

   Since $3^4-5^3=-44=22\cdot(-2)$, $S(1)$ is true.

   Assume $S(k)$ is true for some positive integer k, that is $3^{3k+1}-5^{k+2}=22a$ for some integer a.

   Consider $S(k+1)$: $3^{3k+4}-5^{k+3}=27\cdot 3^{3k+1}-5\cdot 5^{k+2}=27(22a+5^{k+2})-5\cdot 5^{k+2}=22(27a+5^{k+2})$.

   In other words, $S(k+1)$ is also true.

   By the Principle of Mathematical Induction, $S(n)$ is true for all positive integers n.